PART A QUESTIONS WITH ANSWERS

VECTOR CALCULUS

1) Prove that the vector field $\bar{F} = yz\,\bar{i} + xz\,\bar{j} + xy\,\bar{k}$ is irrotational.

Solution:

$$Curl\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \bar{i}[x-x] - \bar{j}[y-y] + \bar{k}[z-z]$$

= 0.

Therefore F is irrotational.

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2) Using divergence theorem, evaluate $\iint \bar{r}\cdot\bar{n}\,dS$ where S is closed surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

By GDT $$\iint_S \bar{r}\cdot\bar{n}\,dS = \iiint_V \nabla\cdot\bar{r}\,dV$$

$$\iiint_V \nabla\cdot\bar{r}\,dV = \iiint_V \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot(x\bar{i} + y\bar{j} + z\bar{k})\,dV$$

$= 3\iiint_V dV = 3\,\frac{4}{3}\pi a^3$ (Since the volume of the sphere with center 'a' is $\frac{4}{3}\pi a^3$)

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3) State Stoke's theorem.

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\bar{n}\,dS$$

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4) State Gauss's divergence theorem.

$$\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V \nabla\cdot\bar{F}\,dV$$

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5) State Green's theorem.

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

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6) Find curl if $\overline{F} = xy\,\overline{i} + yz\,\overline{j} + zx\,\overline{k}$.

Solution:

$$Curl\,\overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$= \overline{i}[0-1] - \overline{j}[z-0] + \overline{k}[0-x]$

$= -y\,\overline{i} + z\,\overline{j} - x\overline{k}$

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7) If $\overline{r}$ is the position vector of the point (x, y, z) prove that $\nabla(r) = \frac{\overline{r}}{r}$

Solution WKT, $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$

$r = |\overline{r}| = x\overline{i} + y\overline{j} + z\overline{k}$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla r = \frac{\partial r}{\partial x}\overline{i} + \frac{\partial r}{\partial y}\overline{j} + \frac{\partial r}{\partial z}\overline{k}$$

$$\nabla r = \frac{x}{r}\overline{i} + \frac{y}{r}\overline{j} + \frac{z}{r}\overline{k}$$

$= \frac{\overline{r}}{r}$

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8) A vector field given by $\overline{F} = (x^2 - y^2 + x)\overline{i} - (2xy + y)\overline{j}$. Show that it is irrotational and hence find its scalar potential. Given $\overline{F} = (x^2 - y^2 + x)\overline{i} - (2xy + y)\overline{j}$

$$Curl\,\overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + x & 2xy + y & 0 \end{vmatrix}$$

$= \overline{i}[0-0] - \overline{j}[0-0] + \overline{k}[2y - 2y]$

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$= \bar{0}$ Therefore it is irrotational.

To find the scalar potential

Given $\bar{F} = \bar{\nabla}\phi$

$$\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$$

$$\frac{\partial\varphi}{\partial x} = (x^2 - y^2 + x),\ \frac{\partial\varphi}{\partial y} = -(2xy + y),\ \frac{\partial\varphi}{\partial z} = 0$$

Integrating w.r.t. to x and y respectively we get

$$\varphi = \int (x^2 - y^2 + x)\,dx$$

$$= \frac{x^3}{3} - y^2x + \frac{x^2}{2} + f(y,z)$$

$$\varphi = \int (x^2 - y^2 + x)dy$$

$$= x^2y - \frac{y^3}{3} + xy + f(x,z)$$

$$= \frac{x^3}{3} - y^2x + \frac{x^2}{2} + x^2y - \frac{y^3}{3} + xy + C$$

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9) The temperature at a point (x, y, z) in a space is given by T(x, y, z) = $x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Solution:

To find the direction such that the temperature becomes cooler we have to find ∇T.

We know that $\nabla T = \frac{\partial T}{\partial x}\bar{i} + \frac{\partial T}{\partial y}\bar{j} + \frac{\partial T}{\partial z}\bar{k}$

$$= 2x\bar{i} + 2y\bar{j} - \bar{k}$$

At the point (1, 1, 2), $\nabla T = 2\bar{i} + 2\bar{j} - \bar{k}$

$\nabla T = 2\bar{i} + 2\bar{j} - \bar{k}$ is the direction in which mosquito must fly such that the temperature get cool faster.

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10) Find $\nabla(\log r)$.

Solution:

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$$

$$\nabla \log r = \frac{\partial(\log r)}{\partial x}\bar{i} + \frac{\partial(\log r)}{\partial y}\bar{j} + \frac{\partial(\log r)}{\partial z}\bar{k}$$

$$=\frac{1}{r}\frac{x}{r}\bar{i}+\frac{1}{r}\frac{y}{r}\bar{j}+\frac{1}{r}\frac{z}{r}\bar{k}$$

$$=\frac{1}{r}\frac{[x\bar{i}+y\bar{j}+z\bar{k}]}{r}$$

$$=\frac{1}{r^2}[\bar{r}]$$

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11) Find a, b, c such that $\bar{F} = (3x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (3x+cy+3z)$ is irrotational.

Solution:

$$Given \quad \nabla\times\bar{F}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = 0$$

$$=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+y+az & bx+2y-z & 3x+cy+3z \end{vmatrix} = 0$$

$$=\bar{i}[c+1]-\bar{j}[3-a]+\bar{k}[b-1]=0$$

$$c+1=0,\ 3-a\ ,\ b-1=0$$

$$c=-1\ ,\ a=3\ ,\ b=1$$

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12) In what direction from (3, 1, -2) is the directional derivative of $\phi=x^2\,y^2\,z^4$ maximum? Find also the magnitude of this maximum.

Solution:

We know that

$$\nabla\varphi=\frac{\partial\varphi}{\partial x}\bar{i}+\frac{\partial\varphi}{\partial y}\bar{j}+\frac{\partial\varphi}{\partial z}\bar{k}$$

$$=2xy^2z^4\bar{i}+2x^2yz^4\bar{j}+4x^2y^2z^3\bar{k}$$

$$\nabla\varphi_{(3,1,-2)}=[2(3)(1)(16)]\bar{i}+[2(9)(1)(16)]\bar{j}+[4(9)(1)(-8)]\bar{k}$$

$$=96\bar{i}+288\bar{j}-288\bar{k}$$

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13) Find α such that $\bar{F} = (3x-2y+z)\bar{i} + (4x+\alpha y-z)\bar{j} + (x-y+2z)\bar{k}$ is solenoid.

Solution:

Given $\overline{F} = (3x-2y+z)\bar{i} + (4x+\alpha y - z)\bar{j} + (x-y+2z)\bar{k}$

Since it is solenoidal $\nabla \cdot \overline{F} = 0$

$$\nabla \cdot \overline{F} = \frac{\partial(3x-2y+)}{\partial x} + \frac{\partial(4x+\alpha y - z)}{\partial y} + \frac{\partial(x-y+2z)}{\partial z} = 0$$

$3 + \alpha + 2 = 0$

$\alpha + 5 = 0$

$\therefore$ **α = -5**

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14) Find a unit vector normal to $x^2 + y^2 + z^2 = 5$ at (0, 1, 2).

Solution:

Given $\varphi = x^2 + y^2 + z^2$

WKT the unit normal $\bar{n} = \frac{\nabla \varphi}{|\nabla \varphi|}$

$$\nabla \varphi = \frac{\partial \varphi}{\partial x}\bar{i} + \frac{\partial \varphi}{\partial y}\bar{j} + \frac{\partial \varphi}{\partial z}\bar{k}$$

$$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\nabla \varphi_{(0,1,2)} = [2(0)]\bar{i} + [2(1)]\bar{j} + [2(2)]\bar{k}$$

$$= 2\bar{j} + 4\bar{k}$$

$$\bar{n} = \frac{2\bar{j} + 4\bar{k}}{|2\bar{j} + 4\bar{k}|}$$

$$= \frac{2\bar{j} + 4\bar{k}}{\sqrt{4+16}} = \frac{2\bar{j} + 4\bar{k}}{\sqrt{20}} = \frac{2\bar{j} + 4\bar{k}}{\sqrt{4 \times 5}} = \frac{\bar{j} + 2\bar{k}}{\sqrt{5}}$$

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15) If $\overline{F} = \overline{\nabla}\phi$, then find $\int_A^B \overline{F} \cdot d\,\bar{r}$

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16) Find the directional derivative of $\phi = x^2 + y^2 + z^2$ at the point (1, 1, 1) in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$

Solution:

We know that Directional derivative = $\nabla\varphi\cdot\dfrac{\bar{a}}{|\bar{a}|}$

$$\nabla\varphi\, at\ (1,1,1) = 2\bar{i}+2\bar{j}+2\bar{k}\ \cdot$$

$$= 2\bar{i}+2\bar{j}+2\bar{k}\cdot\frac{\bar{i}+2\bar{j}+2\bar{k}}{|\bar{i}+2\bar{j}+2\bar{k}|}$$

$$= \frac{2+4+4}{\sqrt{1+4+4}}$$

$$= \frac{10}{3}$$

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17) Define solenoidal vector function. If $\bar{V}=(x+3y)\bar{i}+(y-2z)\bar{j}+(x+2\lambda z)\bar{k}$ is solenoidal, find the value of k.

Solution:

Given $\bar{V}=(x+3y)\bar{i}+(y-2z)\bar{j}+(x+2\lambda z)\bar{k}$

Since it is solenoidal $\nabla\cdot\bar{F}=0$

$$\frac{\partial(x+3y)}{\partial x}+\frac{\partial(y-2z)}{\partial y}+\frac{\partial(x+2\lambda z)}{\partial z}=0$$

$\Rightarrow 1+1+2\lambda = 0$

$\Rightarrow 2+2\lambda = 0$

$\Rightarrow 1+\lambda=0$

$\Rightarrow \lambda= -1$

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18) Find the directional derivative of $\phi = xy^2 + yz^2$ at the point (2, -1, 1) in the direction of the vector $\bar{i}+2\bar{j}+3\bar{k}$

Solution:

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19) Prove that div ($r^n\bar{r}$) = (n+ 3)r^n. Deduce that $r^n\bar{r}$ is solenoidal if and only if n =-3.

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20) Find the angle between the surfaces $x^2+ y^2 + z^2 = 9$ and $z = x^2 + y^2 -3$ at (2, -1, 2).

Solution:

Given $\varphi_1 = x^2 + y^2 - 3$, $\varphi_2 = x^2 + y^2 + z^2 - 9$

$\nabla\varphi_1 = 2x\,\bar{i} + 2y\bar{j} - \bar{k}$, $\nabla\varphi_2 = 2x\,\bar{i} + 2y\bar{j} + 2z\,\bar{k}$

$\nabla\varphi_{1\,at(2,-1,2)} = 4\bar{i} - 2\bar{j} - \bar{k}$, $\nabla\varphi_{2\,at(2,-1,2)} = 4\bar{i} - 2\bar{j} + 4\bar{k}$

$$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}\,,\quad |\nabla\varphi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\cos\theta = \frac{\nabla\varphi_1 \,.\, \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$$\frac{(4\bar{i} - 2y\bar{j} - \bar{k}).(4\bar{i} - 2\bar{j} + 4\bar{k})}{\sqrt{21}\sqrt{36}} = \frac{8}{3\sqrt{21}}$$

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21) If $\bar{F}$ is a vector point function prove that curl (curl $\bar{F}$) $= \nabla(\nabla\cdot F) - \nabla^2 F$.

Given $\nabla\times(\nabla\times\bar{F}) = (\nabla\cdot\bar{F})\nabla - (\nabla\cdot\nabla)\bar{F}$

Since“ $a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$ ”

$= \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}$

=grad (div F) - ∇^2F

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22) Find ∇r^3 where r = $|\bar{r}|$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\nabla r^3 = \sum \bar{i}\frac{\partial}{\partial x}[r^3]$$

$$= \sum \bar{i}\, 3r^2 \frac{\partial r}{\partial x}$$

$$= \sum \bar{i}\, 3r^2 \frac{x}{r}$$

$$= 3r[x\bar{i} + y\bar{j} + z\bar{k}$$

$$= 3r\bar{r}$$

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23) Prove that for any closed surface S, $\iint_S curl\,\bar{F}\cdot\bar{n}\,dS = 0$

WKT Gauss Divergence theorem, “ $\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V \nabla\cdot\bar{F}\,dV$

Where v is the volume of the closed surface s

Since ∇. (Curl F) =0 , we get $\iiint_V \nabla\cdot\bar{F}\,dV = 0$

$$\therefore \iint_S curl\,\bar{F}\cdot\bar{n}\,dS = 0$$

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